Lower and Upper Bound Calculations on the Capacitance of Multiconductor Printed Transmission Line Using the Spectral-Domain Approach and Variational Method

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Abstract—**A unified spectral-domain method is developed for accurate evaluation of the parameters of single and coupled microstripline-type structures containing a number of additional conducting strips with induced and/or zero potentials, located on several interfaces of dielectric layers. The Green's function technique in the spectral domain and the superposition principle for solutions of simple Dirichlet's problems are applied for the first step of the analysis in which a set of algebraic equations is to be derived. Extreme values of two variational functionals are found for estimation of the upper and lower bounds on the line capacitance. Specific computations, carried out for new coupled coplanar lines with additional tuning conductive septums, illustrate the validity and efficiency of the presented method. It has been shown that equalization of the even- and odd-mode phase velocities can be achieved in this structure.**

I. Introduction

THE SPECTRAL-DOMAIN method is commonly used for analyzing the printed line structures. The quasi-TEM approach has been applied to structures with conductors placed either at one [1], [2] or at several [3], [4] interfaces between dielectric layers. However, to date, only the lower bound on capacitance of lines such as those above has been obtained by this technique as a result of using the Green's functions in the Fourier transform domain and under assumption of the charge density distributions on conducting strips expressed in terms of basis functions [1]–[4]. Araki and Naito [5] and Sachse and Sawicki [6], dealing with the potential distribution at the interface with conductors, have calculated the upper bound on capacitance for lines with conductors located only on one interface, i.e., for microstrip and coplanar lines.

There is no published spectral-domain method for a two-sided estimation on capacitance of the line composed of a number of conducting strips located on several interfaces of dielectric layers. In this paper, a generalized formulation of the quasi-TEM spectral-domain method is described to complete that deficiency. This formulation leads to various sets of algebraic equations giving the relations between the Fourier transforms of potential and charge density distributions at every interface with conducting strips. These sets of equations are found easily and interpreted formally solving simple Dirichlet boundary problems by means of Green's function technique in the spectral domain. It is shown that any form of derived equations can be used to compute the line capacitance by applying the Galerkin's method and Parseval's identity. Two specific forms of these equations are utilized for construction of two functionals, extreme values of which allow one to estimate the lower and upper bounds on the capacitance.

Some numerical results are presented for the new coupled coplanar lines with additional tuning conductive septums introduced on the bottom side of the substrate.

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II. Analysis

A. Formulation

Fig. 1 shows the cross section of the single or coupled printed transmission lines composed of n dielectric layers and N $(N < n)$ interfaces with conducting strips. It is helpful to distinguish the so-called main interface $y = h_{cM}$ on which the main strips of the line are placed (single or coupled). The conductors located on the auxiliary interfaces, $y = h_{c1}, h_{c2}, \cdots, h_{cN}$, $y \neq h_{cM}$, are assumed to have induced or zero potentials. To simplify the analysis, symmetry of the structure with respect to the y axis is also assumed.

The above assumptions do not restrict the general validity of the presented method, but permit one to analyze in a straightforward manner the most commonly used transmission-line structures in which both the tuning septums [3], [4], [7] or induced potential conductors [8], [9] are employed. This method can be adopted easily for analyzing a number of microstrip-type structures with several main interfaces, as well as for the ones without structural symmetry.

In order to compute the capacitance of the line shown in Fig. 1, one needs to solve Laplace's equation in the right-

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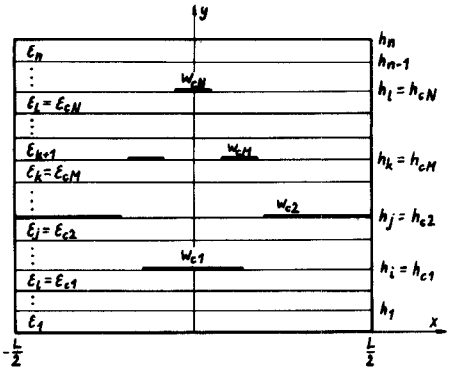

Fig. 1. The cross section of stripline containing n dielectric layers and N interfaces with conductors.

hand half cross section subject to appropriate boundary conditions. It is convenient to introduce the discrete Fourier transform of the potential

$$\tilde{\varphi}(k_m, y) = \frac{4}{L}\int_0^{L/2} \varphi(x, y)\begin{Bmatrix}\cos k_m^{(e)} x \\ \sin k_m^{(o)} x\end{Bmatrix} dx \qquad (1)$$

where, for the even mode, $k_m^{(e)} = [(2m-1)/L]\pi$, and, for the odd mode, $k_m^{(o)} = (2m/L)\pi$, $m = 1, 2, \cdots$. Using (1), the partial differential equation transforms to an ordinary differential equation

$$\left(\frac{d^2}{dy^2} - k_m^2\right)\tilde{\varphi}(k_m, y) = 0. \qquad (2)$$

Solution of (2) in the jth layer takes the form

$$\tilde{\varphi}_j(k_m, y) = C_j(k_m)\sinh k_m y + D_j(k_m)\cosh k_m y, \quad j = 1, \cdots, n \qquad (3)$$

where $C_j(k_m)$ and $D_j(k_m)$ can be derived from the boundary conditions assumed at the interfaces with conducting strips and from the continuity conditions.

It is worth stressing that, in this analysis, uniform as well as mixed boundary conditions can be assumed alternatively. In the case of uniform boundary conditions, the charge density or potential distributions at every interface with conducting strips are to be settled. Using mixed boundary conditions, the charge density distributions at some interfaces, and the potential distributions at the others, are to be assumed. For any system of boundary conditons, one can derive the transforms of quantities dual to those assumed at N interfaces with conducting strips.

If at a certain interface $y = h_{cl}$ the charge density distribution is assumed, the transform of dual quantity (potential distribution) at this interface is, according to (3), equal to

$$\tilde{\varphi}(k_m, h_{cl}) = C_1(k_m)\sinh k_m h_{cl} + D_l(k_m)\cosh k_m h_{cl} \qquad (4)$$

and, reciprocally, if the potential distribution is assumed as the boundary condition at this interface, the transform of the charge density distribution can be expressed as

$$\tilde{\rho}(k_m, h_{cl}) = \epsilon_{cl}\frac{\partial}{\partial y}\tilde{\varphi}(k_m, y)\Big|_{y = h_{cl} - 0} - \epsilon_{cl+1}\frac{\partial}{\partial y}\tilde{\varphi}(k_m, y)\Big|_{y = h_{cl} + 0} \qquad (5)$$

where $\tilde{\varphi}(k_m, y)$ is given by (3).

Application of (4) and (5) for each interface with conductors, $y = h_{cl}$, $l = 1, \cdots, N$, leads to the following generalized $N \times N$ matrix equation:

$$[\tilde{r}(k_m)] = [\tilde{R}(k_m)][\tilde{s}(k_m)]. \qquad (6)$$

The one column matrices $[\tilde{s}(k_m)]$ and $[\tilde{r}(k_m)]$ in (6) will be further named the source and reaction vectors, respectively. The source vector is composed of the Fourier transforms of charge density and/or potential distributions at interfaces with conductors. The reaction vector is created by quantities dual to those creating the source vector. The matrix $[\tilde{R}(k_m)]$ in (6) depends on the form of the source vector and on the structure of the line.

The source vector can take two particular forms. If this vector includes only transforms of the charge density distributions, (6) can be written as follows:

$$[\tilde{\varphi}(k_m)] = [\tilde{G}(k_m)][\tilde{\rho}(k_m)]. \qquad (7)$$

And in the dual case, if there are only transforms of the potential distributions, one has

$$[\tilde{\rho}(k_m)] = [\tilde{F}(k_m)][\tilde{\varphi}(k_m)]. \qquad (8)$$

Notice that in the case when the matrices $[\tilde{G}(k_m)]$ and $[\tilde{F}(k_m)]$ are derived for the same structure of the line, the following relationship is true:

$$[\tilde{G}(k_m)] = [\tilde{F}(k_m)]^{-1}. \qquad (9)$$

Equation (7) is already known from [4]. Equation (8) has been obtained in [6] for the line structure with just one interface with conductors. Equation (8), as well as its more general form (6) (for the source vector with arbitrary assumed elements), and also the identity (9), can be used for more general analysis of multiconductor and multilayered printed transmission lines. As it will be proven in Section I-D, the use of (7) and (8) leads to the two-sided estimation on capacitance of the line.

B. Derivation and Interpretation of the $[\tilde{R}(k_m)]$ Matrix Elements

In Section II-A, elements of the matrix $[\tilde{R}(k_m)]$ have been derived using the boundary and continuity conditions at the dielectric interfaces. However, if the number of layers is large, that method is not efficient because of the necessity for the analytical solution of a large system of algebraic equations. In this instance, a more efficient original method can be applied. In this method, elements of the $[\tilde{R}(k_m)]$ matrix are obtained by solving a number of Dirichlet boundary problems in the spectral domain. The method can also give a formal interpretation of the $[\tilde{R}(k_m)]$ matrix elements. To date, only the elements of matrix

$[\tilde{G}(k_m)]$ have their clear interpretation as transforms of Green's functions [4].

First of all, let us recall the well-known solutions for two particular cases of the boundary problems in the space domain. In both cases, a dielectrically nonhomogeneous region R is considered. If this region is shielded, and includes a set of conductors with a distribution of charge density $\rho(x, y)$, the Dirichlet's problem for Poisson's equation is to be solved. Solution of this problem takes the form [10]

$$\varphi(x, y) = \oint_{C_s} \rho(x_s, y_s) G(x, y; x_s, y_s) dC_s \qquad (10)$$

where $G(x, y; x_s, y_s)$ is Green's function and C_s means the contour surrounding each conductor. If the region R is sourceless, i.e., $\rho(x, y) = 0$, and at a certain part of the edge ∂R_i of the subregion R_i the potential distribution $\varphi_{Ei}(x, y)$ is settled, and at the rest of the R region the edge potential is equal to zero, solution of the Dirichlet's problem for Laplace's equation is [10]

$$\varphi(x, y) = -\epsilon_i \int_{\partial R_i} \varphi_{Ei}(x_s, y_s) \frac{\partial G(x, y; x_s, y_s)}{\partial n_s} dC_s \qquad (11)$$

where $\vec{n}_s$ is the unit vector directed outside the subregion R_i, and ϵ_i is the dielectric permittivity of the media filling this subregion.

Consider now the region composed of a planar arrangement of n dielectric layers and N interfaces with conductors (as in Fig. 1). In this case, one can employ the Fourier transform defined by (1) to (10) and (11). Because (10) and (11) are the convolution integrals, the following expressions in the spectral domain can be obtained:

$$\tilde{\varphi}(k_m, y) = \tilde{\rho}(k_m, y_s) \tilde{G}(k_m, y, y_s) \qquad (12)$$

$$\tilde{\varphi}(k_m, y) = -\epsilon_i \tilde{\varphi}_{Ei}(k_m, y_s) \frac{\partial \tilde{G}(k_m, y, y_s)}{\partial n_s} \qquad (13)$$

where the quantities signed with a wave line are Fourier transforms of the corresponding quantities in (10) and (11). Expressions (12) and (13) will now be used for derivation and interpretation of the $[\tilde{R}(k_m)]$ matrix elements for some specified forms of the source vector.

Let us assume in the first case the charge density distributions at every interface with conductors (the source vector is $[\tilde{\rho}(k_m)]$). In this case, the matrix $[\tilde{G}(k_m)]$ can be found by solving the Dirichlet boundary problems for Poisson's equation as shown in Fig. 2 (the interfaces with conductors are signed with dashed lines). Using (12), the following relation betwen the Fourier transform of potential distribution at the ith interface and the transform of the charge density distribution at the jth interface is obtained:

$$\tilde{\varphi}_{ij}(k_m) = \tilde{G}_{ij}(k_m) \tilde{\rho}_j(k_m) \qquad (14)$$

in which $\tilde{G}_{ij}(k_m)$ is the transform of the Green's function.

In the second case, the source vector is assumed to be made up of the transforms of potential distributions at every interface (the matrix $[\tilde{F}(k_m)]$ is to be found). This

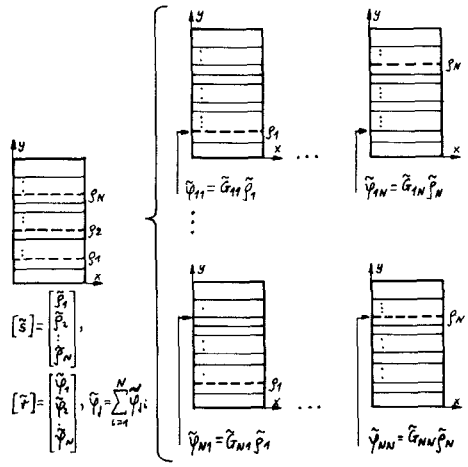

Fig. 2. Superposition of simple Dirichlet's problems for Poisson's equation (the source vector composed of charge density distributions at every interface).

boundary problem can be decomposed on many simple Dirichlet boundary problems for Laplace's equation in specified subregions on the edge of which the potential functions are assumed (Fig. 3). Notice that the elements $\tilde{F}_{lk}(k_m)$ are equal to zero when l and k are differing more than 1. So, $[\tilde{F}(k_m)]$ is a tridiagonal matrix.

Elements $\tilde{F}_{ll}(k_m)$ at the main diagonal are derived from solutions of Laplace's equation in two subregions adjacent to the interface $y = h_{cl}$. These subregions are bounded from the other side by the nearest interfaces with conductors. Using (13) for the mentioned subregions and then calculating the transforms of the component of the electric induction vector normal to the interface $y = h_{cl}$, one obtains the following expression for the elements $\tilde{F}_{ll}(k_m)$:

$$\tilde{F}_{ll}(k_m) = \sum_{j=0}^{1} \epsilon_{cl+j} \vec{n}_j \cdot \left\{ -\vec{a}_y \frac{\partial}{\partial y} \left[-\epsilon_{cl+j} \frac{\partial \tilde{G}_{cl+j}(k_m, y, y_s)}{\partial n_{sj}} \bigg|_{y_s = h_{cl}} \right] \bigg|_{y = h_{cl}} \right\} \qquad (15)$$

where

ϵ_{cl+j} dielectric permittivity of the layers nearest to the interface $y = h_{cl}$ (for the lower layer $j = 0$ and for the upper one $j = 1$),

$\tilde{G}_{cl+j}(k_m, y, y_s)$ transforms of the Green's functions for the lower ($j = 0$, $y, y_s \leqslant h_{cl}$), and upper ($j = 1$, $y, y_s \geqslant h_{cl}$) subregions,

$\vec{a}_y$ unit vector along the y axis,

$\vec{n}_{sj}$ unit vector directed outside the subregions (for $j = 0$ $\vec{n}_{s0} = \vec{a}_y$, for $j = 1$ $\vec{n}_{s1} = -\vec{a}_y$),

$\vec{n}_j$ normal unit vector directed outside the conductors located on the interface $y = h_{cl}$ (for $j = 0$ $\vec{n}_0 = -\vec{a}_y$, for $j = 1$ $\vec{n}_1 = \vec{a}_y$).

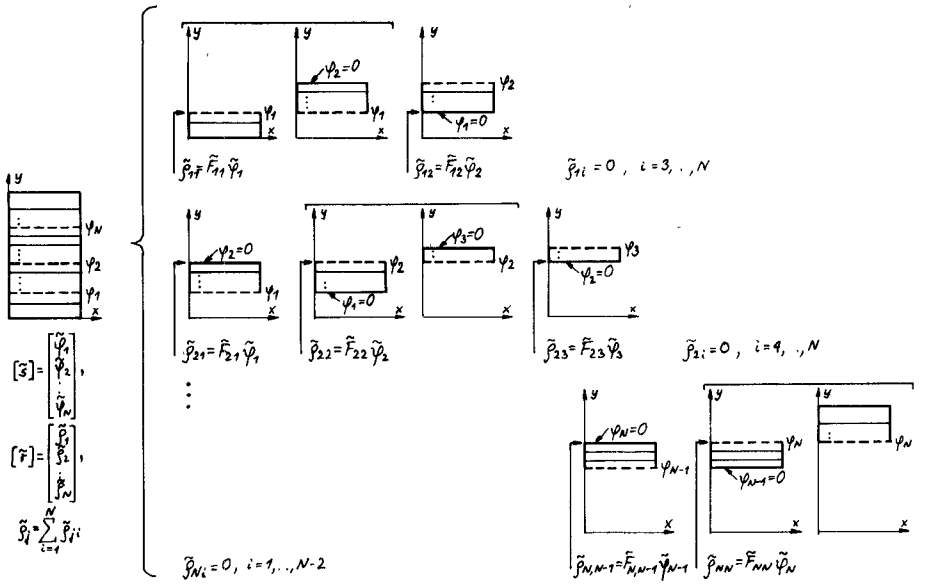

Fig. 3. Superposition of simple Dirichlet's problems for Laplace's equation (the source vector composed of potential distributions at every interface).

Elements $\tilde{F}_{l,l+1}(k_m)$, $l=1,\cdots,N-1$, and $\tilde{F}_{l,l-1}(k_m)$, $l=2,\cdots,N$, can be derived in a similar way as the elements expressed by (15). There is however, a difference: the source (potential distribution) is now settled at the interface $y=h_{c(l+1)}$ (or $y=h_{c(l-1)}$) and reaction (charge density distribution) is to be found at the interface $y=h_{cl}$. Finally, element $\tilde{F}_{l,\,l+1}(k_m)$ can be expressed in the form

$$\tilde{F}_{l,\,l+1}(k_m)=\epsilon_{cl+1}\vec{n}\left\{-\vec{a}_y\frac{\partial}{\partial_y}\left[-\epsilon_{c(l+1)}\frac{\partial\tilde{G}_{cl+1}(k_m,y,y_s)}{\partial n_s}\bigg|_{y_s=h_{c(l+1)}}\right]\bigg|_{y=h_{cl}}\right\} \tag{16}$$

where $\tilde{G}_{cl+1}(\cdots)$ is the transform of the Green's function for the subregion bounded by the coordinates $y=h_{cl}$ and $y=h_{c(l+1)}$, and $\vec{n}=\vec{n}_s=\vec{a}_y$. Element $\tilde{F}_{l,\,l-1}(k_m)$ can be derived similarly.

Notice that the transforms of Green's functions in (15) and (16) are defined for simple subregions (usually composed of the single- or double-layered dielectric media). Tridiagonality of the matrix $[\tilde{F}(k_m)]$ is an advantageous feature of this matrix because it certainly reduces the complexity of calculations. Therefore, the matrix $[\tilde{F}(k_m)]$ can be used as a fundamental matrix for the two-sided estimation of capacitance of the line (the second matrix needed for this purpose can be easily found numerically from (9)).

Equations (15) and (16) provide the interpretation of the $[\tilde{F}(k_m)]$ elements. These elements are proportional to the second mixed derivatives of the transforms of the Green's functions, defined for the particular subregions, with respect to the coordinates y_s and y.

In the case of mixed boundary conditions, the manner of decomposition of the line cross section depends on the form of the source vector. As an example, consider a simple mixed boundary problem illustrated in Fig. 4. There are two problems qualitatively new as compared with those presented previously. These problems require deriving the elements $\tilde{R}_{12}(k_m)$, $\tilde{R}_{13}(k_m)$ and, respectively, $\tilde{R}_{21}(k_m)$, $\tilde{R}_{31}(k_m)$.

The first problem is the Dirichlet boundary problem for Poisson's equation in which one should calculate additionally the transform of the charge density distribution at the edge (at $y=h_1$). For sources placed respectively at the interfaces $y=h_2$ and $y=h_3$, one can derive, using (12), the following identities:

$$\tilde{\rho}_{1j}(k_m)=\underbrace{\epsilon_2\vec{n}\left[-\vec{a}_y\frac{\partial}{\partial y}\tilde{G}(k_m,y,y_s=h_j)\bigg|_{y=h_1}\right]}_{\tilde{R}_{1j}(k_m)}\tilde{\rho}_j(k_m),\qquad j=2,3. \tag{17}$$

The second problem is a simple Dirichlet boundary problem for Laplace's equation. Using (13), one obtains the following expressions for the elements $\tilde{R}_{21}(k_m)$ and $\tilde{R}_{31}(k_m)$:

$$\tilde{R}_{j1}(k_m)=-\epsilon_2\frac{\partial G(k_m,y=h_j,y_s)}{\partial n_s}\bigg|_{y_s=h_1},\qquad j=2,3. \tag{18}$$

$\tilde{G}(k_m,y,y_s)$ in (17) and (18) is the transform of the Green's function defined for the subregion bounded by the coordinates $y=h_1$ and $y=h_4$.

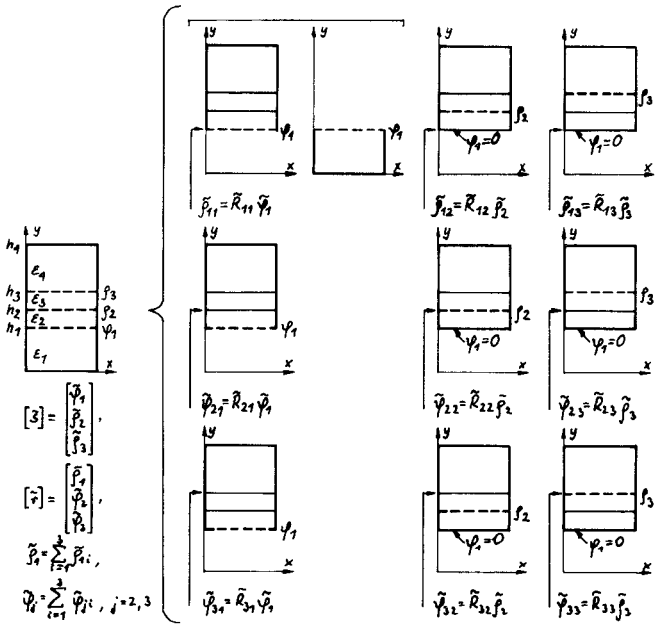

Fig. 4. Superposition of simple Dirichlet boundary problems for the mixed source vector.

In a similar manner, one can analyze any line structure "sourced" with any source vector. In the presented method, it is required to derive the transforms of the Green's functions for simple subregions of the line cross section. A fundamental collection of these transforms can be derived easily and utilized for all the cases of the source vectors to be used.

C. The Galerkin's Procedure

To solve matrix equation (6), an unknown vector $[\tilde{s}(k_m)]$ is expanded as

$$\tilde{s}_l(k_m) = \sum_{i=1}^{K_l} c_l^i \tilde{s}_l^i(k_m), \qquad l=1,\cdots,N \tag{19}$$

where c_l^i, $i=1,\cdots,K_l$, $l=1,\cdots,N$, are unknown coefficients. Substituting (19) into (6) and taking the inner products of the resultant equations with $\tilde{s}_l^i(k_m)$, $i=1,\cdots,K_l$, $l=1,\cdots,N$, the following set of coupled linear equations for c_l^i is obtained:

$$[A][c]=[B] \tag{20}$$

where

$$[A]=\begin{bmatrix}[A_{11}] & \cdots & [A_{1N}]\\ \vdots & & \\ [A_{N1}] & & [A_{NN}]\end{bmatrix} \quad [B]=\begin{bmatrix}[B_1]\\ \vdots\\ [B_N]\end{bmatrix}$$

$$A_{kl}^{ij}=\langle \tilde{R}_{kl}(k_m)\tilde{s}_k^i(k_m),\ \tilde{s}_l^j(k_m)\rangle$$

$$B_l^i=\langle \tilde{r}_l(k_m),\ \tilde{s}_l^i(k_m)\rangle$$

and $\langle\cdots,\cdots\rangle$ means the inner product expressed as follows:

$$\langle \tilde{f}(k_m),\ \tilde{g}(k_m)\rangle = \frac{L}{2}\sum_{m=1}^{\infty}\tilde{f}(k_m)\tilde{g}(k_m) \tag{21}$$

where $\tilde{f}(k_m)$ and $\tilde{g}(k_m)$ are transforms of functions $f(x)$ and $g(x)$.

An unknown one-column matrix $[B]$ in (20) can be defined by virtue of Parseval's identity. Because the conductors located on the auxiliary interfaces have induced or zero potential, the elements of the matrix $[B]$ assigned to these interfaces are equal to zero. So, the matrix $[B]$ takes two forms that depend on the boundary conditions assumed at the main interface. If, for this interface, the element of the source vector is the transform of the charge density distribution, one should assume the potential V_M of the main strip. Then, the elements of matrix $[B]$ take the form

$$B_l^i=\begin{cases} V_M\int_{w_cM}\rho_M^i(x)\,dx, & l=M,\ i=1,\cdots,K_M\\ 0, & l=1,\cdots,N,\ i\neq M\end{cases} \tag{22}$$

where $\rho_M^i(x)$ is the ith basis function approximating the charge density distribution on the strip w_{cM}.

If the main element of the source vector is the transform of the potential distribution, one should assume the value Q_M of the total charge concentrated on the main strip. Then

$$B_l^i=\begin{cases} Q_M\varphi_M^i(x)|_{x\in w_{cM}}, & l=M,\ i=1,\cdots,K_M\\ 0, & l=1,\cdots,N,\ i\neq M\end{cases} \tag{23}$$

where $\varphi_M^i(x)|_{x\in w_{cM}}$ is the potential of the main strip "generated" by the ith basis function approximating the distribution of the potential at the main interface.

The capacitance of the line can be calculated as a ratio of the total charge and the potential of the main strip. So, in the first case, the capacitance of the line equals

$$C_V=\frac{1}{V_M}\sum_{i=1}^{K_M}c_M^i\int_{w_{cM}}\rho_M^i(x)\,dx \tag{24}$$

where coefficients c_M^l are solutions of the matrix equation (20) in which the matrix $[B]$ has been replaced by (22).

In the dual case, the capacitance of the line is

$$C_Q = Q_M \left[\sum_{l=1}^{K_M} c_M^l \varphi_M^l(x) \Big|_{x \epsilon w_{cM}} \right]^{-1} \tag{25}$$

where c_M^l are solutions of (20) in which one should substitute (23) for the matrix $[B]$.

D. Two-Sided Estimation on the Line Capacitance

Let us introduce the following functional:

$$F = 2W_M - \frac{1}{2} \sum_{l=1}^{N} \int_{-L/2}^{L/2} \varphi_l(x) \rho_l(x)\, dx \tag{26}$$

where $2W_M = \int_{-L/2}^{L/2} \varphi_M(x) \rho_M(x)\, dx$ is the term which the stationary value equals twice the energy stored in the electric field per unit length. As it will be shown, this functional can be applied for two-sided estimation on the line capacitance.

Examine first the case when the potential of the main strip is fixed and equals V_M. Functions $\rho_l(x)$, $l = 1, \cdots, N$, approximating the charge density distributions on every strip conductor, are to be searched for. Using the Parseval's identity for each integral standing below the summation sign in (26) and, according to (7), replacing each transform of the potential function by the product of $\tilde{G}_{kl}(k_m)$ and $\tilde{\rho}_k(k_m)$, one obtains the functional F_V as follows:

$$F_V = V_M \int_{w_{cM}} \rho_M(x)\, dx - \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} \langle \tilde{G}_{kl}(k_m) \tilde{\rho}_k(k_m), \tilde{\rho}_l(k_m) \rangle. \tag{27}$$

Quite similarly, the expression for the functional F_Q (assuming the total charge Q_M concentrated on the main strip and using (8)) can be derived

$$F_Q = Q_M \varphi_M(x)|_{x \epsilon w_{cM}} - \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} \langle \tilde{F}_{kl}(k_m) \tilde{\varphi}_k(k_m), \tilde{\varphi}_l(k_m) \rangle. \tag{28}$$

In this case, functions $\varphi_l(x)$, $l = 1, \cdots, N$, approximating the potential distributions at every interface with conductors, are to be searched for.

In order to find the extrema of the functionals F_V and F_Q, the Ritz method can be applied. This means that functions $\rho_l(x)$ and $\varphi_l(x)$, $l = 1, \cdots, N$, in (27) and (28) should be expanded in terms of basis functions expressed as in (19), and the first derivatives of F_V and F_Q with respect to the unknown coefficients should be calculated and equated to zero. As a result, two sets of linear algebraic equations just like (20) are obtained, in which individual elements derived separately for the functionals F_V and F_Q take the forms, respectively,

$$A_{kl}^{ij} = \langle \tilde{G}_{kl}(k_m) \tilde{\rho}_k^i(k_m), \tilde{\rho}_l^j(k_m) \rangle \tag{29}$$

$$B_l^i = \begin{cases} V_M \int_{w_{cM}} \rho_M^i(x)\, dx, & l = M \\ 0, & l \neq M \end{cases}$$

and

$$A_{kl}^{ij} = \langle \tilde{F}_{kl}(k_m) \tilde{\varphi}_k^i(k_m), \tilde{\varphi}_l^j(k_m) \rangle \tag{30}$$

$$B_l^i = \begin{cases} Q_M \varphi_M^i(x)|_{x \epsilon w_{cM}}, & l = M \\ 0, & l \neq M. \end{cases}$$

Solutions of (20) with (29) and (30) allow for computation of the extrema values F_{Ve}, F_{Qe} of the functionals F_V and F_Q. These extrema values are equal approximately to the energy stored per unit length in the line. Hence, the capacitance value can be estimated from the following formulas:

$$C_{Ve} = \frac{2F_{Ve}}{V_M^2} \tag{31}$$

$$C_{Qe} = \frac{Q_M^2}{2F_{Qe}}. \tag{32}$$

In order to define the nature of the extrema of functionals F_V and F_Q, one should calculate their second variations and, according to [11], utilize the following properties of the matrices $[\tilde{G}(k_m)]$ and $[\tilde{F}(k_m)]$:

$$\sum_{k=1}^{N} \sum_{l=1}^{N} \tilde{G}_{kl}(k_m) > 0,$$

$$\sum_{k=1}^{N} \sum_{l=1}^{N} \tilde{F}_{kl}(k_m) > 0, \qquad m = 1, 2, \cdots. \tag{33}$$

It can be shown that functionals F_V and F_Q reach their maxima for the correct charge density and potential distributions, respectively. Thus, approximate values F_{Ve} and F_{Qe} are smaller than the exact ones. The capacitance value calculated from (31) is smaller, and calculated from (32) is larger than the exact one. The average value of capacitance

$$C = \tfrac{1}{2}(C_{Ve} + C_{Qe}) \tag{34}$$

estimates the exact value with an error smaller than

$$\delta_C = \frac{C_{Qe} - C_{Ve}}{C_{Qe} + C_{Ve}}. \tag{35}$$

Once the line capacitance C_ϵ and C_0 are evaluated for nonhomogeneous (layered) and homogeneous (air-loaded) dielectric media, the characteristic impedance Z_c and the effective dielectric constant ϵ_{eff} can be obtained as follows:

$$Z_c = \frac{1}{v_c} (C_\epsilon C_0)^{-1/2} \tag{36}$$

$$\epsilon_{\text{eff}} = \frac{C_\epsilon}{C_0} \tag{37}$$

where v_c is the light velocity in free vacuum space. Because of the form of (36), it is obvious that the characteristic impedance value is bounded from the reverse side rather than the capacitance value.

Comparing (20), (6), (7), (8), (22), and (23) to (29) and (30), it can be noticed that using both the Galerkin's and Ritz's methods, the same sets of algebraic equations are obtained for the cases when the charge density or potential distributions are approximated at every interface. Thus, the capacitance values calculated from (24) and (31) and also from (25) and (32) are equal.

If mixed boundary conditions are assumed at interfaces with conductors, it is impossible to determine *a priori* from which side the estimation of the line capacitance is taking place. This problem is discussed in the next section.

III. Numerical Results

Modified coupled coplanar lines for which specific computations have been carried out are shown in Fig. 5. This structure can be useful for microwave integrated circuits with a view to improve parameters of various passive components. It is also compatible with the conductor-backed coplanar line, proposed recently by Shih and Itoh [12] for monolithic microwave integrated circuits.

In the computations, the unified system of basis functions has been used. This system is based on the Chebyshev polynomials of the first and second kinds, $T_i(X)$ and $U_i(X)$, $i=0, 1, 2, \cdots$, respectively, weighed by an "edge condition" term $\sqrt{1-X^2}$, where X is the x coordinate normalized to the half of a distance on which the approximation is taking place. The system incorporates a singular behavior of the charge density and electric-field distributions at the conductor edges and also their nonsymmetrical distributions.

In Fig. 6, convergence of the upper and lower bounds on the even- and odd-mode impedances versus a number of terms of basis functions at both interfaces is shown for two different structural parameters.

In the first case (Fig. 6(a)), the coplanar line ground planes had been removed so far away that our computations might be compared with the results for coupled suspended microstrip lines with tuning septums presented by Itoh and Hebert in [3]. Some differences, especially for the odd-mode, are visible. They result from the fact that Itoh and Hebert used only symmetrical terms of basis functions for approximation of the charge density distribution on coupled strips, whereas the physically existing distributions are nonsymmetrical. In our computations, both symmetrical and nonsymmetrical terms have been used; hence, the results are better.

Because of wide slits between the conductors in Fig. 6(a), the potential distributions are approximated worse than the charge density distributions, and the lower bound on the impedance converges to the exact value more slowly than the upper one. In Fig. 6(b), the results shown for the dimensions of the structure chosen by this means that the charge density distributions are approximated worse and the upper bound converges slowly.

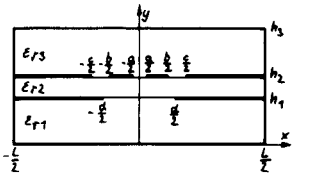

Fig. 5. Modified coupled coplanar lines.

The numerical results obtained for mixed boundary conditions, i.e., for potential distribution settled at the additional ground plane and charge density distributions settled at the plane of coplanar lines, are depicted in Fig. 6 by dashed lines. Based on these results, it can be concluded that, in the case of mixed boundary conditions, the side from which the solution converges to the exact impedance value is determined by the exactness of approximation of the physical quantities at the individual interfaces. If at some interface the charge density distribution is approximated worse than the potential distributions at the remaining interfaces, the upper bound on impedance is obtained. And inversely, if at some interface, the approximation of the potential distribution is the worst, the lower bound on impedance is calculated. One can ensure the best accuracy of the computations in the cases if the charge density distributions are approximated at the planes on which narrow strips are located and the potential distributions at the planes where there are wide strips and narrow slits between them.

The "overlap" effect of some individual solutions can be observed in Fig. 6. This effect appears when the charge density or potential distributions are well approximated by an applied set of basis functions and the errors caused by the truncation of the series used in (20) dominates. A number of harmonic terms in (20) has been fixed individually by the computer program to truncate each series with the same error, and was not smaller than 100.

In Fig. 7, the characteristic impedances and effective dielectric constants for even- and odd-modes versus the slit width in the additional ground plane normalized to the slit width in the main ground plane are shown. These characteristics have been obtained by calculating the upper and lower bounds on impedance and applying $2+6$ terms of basis functions when the charge density distributions were aprroximated, and only $1+3$ terms ($1+2$ for the odd mode) for approximation of the potential distributions. It was estimated that the total error caused by approximation inaccuracy and by truncation errors does not exceed 0.5 percent. On can observe in Fig. 7 that equalization of the even- and odd-mode effective dielectric constants can be achieved. A very important feature of the proposed structure is that it can permit one to design microwave integrated components of higher quality (for example, high-directivity directional couplers).

IV. Conclusions

We have presented a unified quasi-TEM spectral-domain method which is able to estimate the lower and upper bounds on capacitance of multilayer and multiconductor

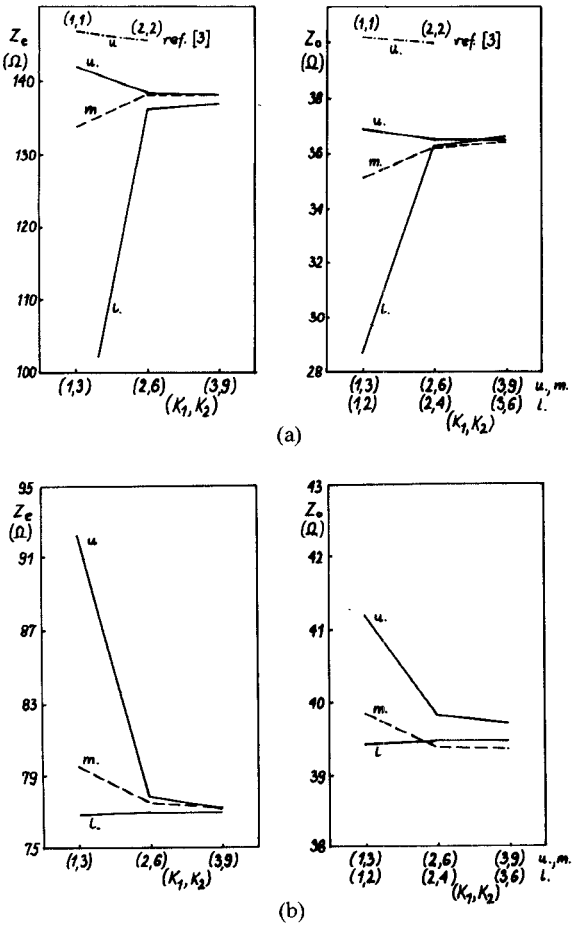

Fig. 6. The upper (u.) and lower (l.) bound on the even(e)- and odd(o)-mode impedances and the results for mixed boundary conditions (m.) versus a number (K_1, K_2) of terms of basis functions at interfaces $y = h_1$ and $y = h_2$, respectively. (a) $\epsilon_{r1} = \epsilon_{r3} = 1$, $\epsilon_{r2} \doteq 9.6$, $h_2 - h_1 = h$, $(h_3 - h_2)/h = h_1/h = 10.0$, $L/h = 20.0$, $a/h = 0.3$, $b/h = 2.7$, $c/h = 18.0$, $d/h = 6.0$. (b) $b/h = 1.5$, $c/h = 2.5$, $d/h = 1.0$.

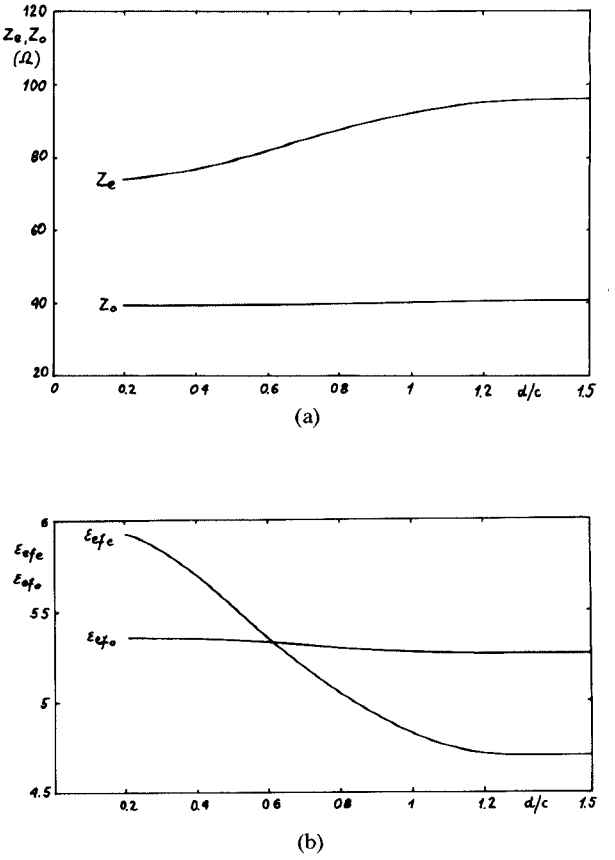

Fig. 7. (a) Characteristic impedances and (b) effective dielectric constants for even(e)- and odd(o)-modes for modified coupled coplanar lines versus normalized slit width; dimensions of the structure are as in Fig. 6(b).

printed lines. It has been shown that the approximation of the charge density distributions at every interface with conductors leads to the lower bound and approximation of the potential distributions to the upper bound. If mixed boundary conditions are assumed, the side of estimation is determined by exactness of the approximation of the charge density or potential distributions at the individual interfaces. Choosing mixed boundary conditions, one may, in particular cases, increase the accuracy of the capacitance calculations.

Numerical results have been given for modified coupled coplanar lines backed by tuning conductive septums. It has been shown that equalization of the even- and odd-mode propagation constants can be achieved in this structure. The wider range of characteristics for these lines, also including the dispersion effect, and measured results for high-directivity couplers built using these lines, will be published in a separate paper.

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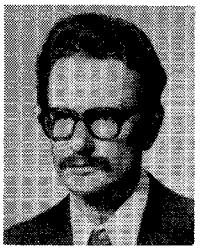

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